Full

$$\underset{m\times n}{A}=\underset{m\times m}{U}\underset{m\times n}{\Sigma}\underset{n\times n}{V^T}=\begin{bmatrix}U_r & U_{m-r}\end{bmatrix}\begin{bmatrix}\Sigma_r & 0_{r\times(n-r)}\\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)}\end{bmatrix}\begin{bmatrix}V_r^T\\ V_{n-r}^T\end{bmatrix}$$

Compact & Outer Product Form

$$A=U_r\Sigma_rV_r^T=\sum_{i=1}^{r}\sigma_i\vec{u}_i\vec{v}_i^T$$

Methods to find the SVD

① Pick A^TA or AA^T (whichever has smaller dimensions)

CASE: A^TA

② Find $\lambda_1\cdots\lambda_n$ of A^TA & order them s.t $\lambda_1\geq\cdots\lambda_r>0$ & $\lambda_{r+1}=\cdots=\lambda_n=0$

③ Find n orthonormal eigenvectors $\vec{v}_i$ s.t. $A^TA\vec{v}_i=\lambda_i\vec{v}_i$ for $i=1,\ldots,n$

④ Define $\sigma_i=\sqrt{\lambda_i}$ for $i=1,\ldots,\min\{m,n\}$

⑤ Find orthonormal vectors $\vec{u}_1,\ldots,\vec{u}_m$ obtaining $\vec{u}_1,\ldots,\vec{u}_r$ by the equation $\vec{u}_i=\frac{A\vec{v}_i}{\sigma_i}$ for $i=1,\ldots,r$

and $\vec{u}_{r+1},\ldots,\vec{u}_m$ via G-S

CASE: AA^T

② Find $\lambda_1\cdots\lambda_m$ of AA^T & order s.t $\lambda_1\geq\cdots\lambda_r>0$ & $\lambda_{r+1}=\cdots=\lambda_m=0$

③ Find m orthonormal eigenvectors $\vec{u}_i$ s.t. $AA^T\vec{u}_i=\lambda_i\vec{u}_i$ for $i=1,\ldots,m$

④ → (do the same)

⑤ Find orthonormal vectors $\vec{v}_1,\ldots\vec{v}_n$ obtaining $\vec{v}_1\cdots\vec{v}_r$ by the eqn $\vec{v}_i=\frac{A\vec{u}_i}{\sigma_i}$ for $i=1,\ldots,r$

and finding $\vec{v}_{r+1}\cdots\vec{v}_n$ via G-S

$m\times m$

$\underset{n\times n}{V}$: $\vec{v}_1\perp\vec{v}_2\perp\ldots\perp\vec{v}_n$ in $\mathbb{R}^n$

$\underset{m\times n}{\Sigma}$: $\underset{r\times r}{\Sigma_r}$ in upper left, 0 elsewhere

- cols of U are orthonormal e-vecs of AA^T
- cols of V are orthonormal e-vecs of A^TA
- diagonal entries in Σ are square roots of eigenvalues of AA^T or A^TA

$m\times r$ $n\times r$ $\vec{u}_1\cdots\vec{u}_r$ & $\vec{v}_1\cdots\vec{v}_r$ of U & V.

$\underset{r\times r}{\Sigma_r}$: largest r singular values $\sigma_1\geq\cdots\geq\sigma_r>0$ of A

$\underset{m\times(m-r)}{U_{m-r}}$: last m−r orthonormal cols $\vec{u}_{r+1}\cdots\vec{u}_m$ of U

$\underset{n\times(n-r)}{V_{n-r}}$: last n−r orthonormal cols $\vec{v}_{r+1}\cdots\vec{v}_n$ of V

- $\text{col}(U_R)=\text{span}(\vec{u}_1\cdots\vec{u}_R)=\text{col}(A)$
- $\text{col}(U_{m-r})=\text{span}(\vec{u}_{r+1}\cdots\vec{u}_m)\perp\text{col}(A)^2$
- $\text{col}(V_r)=\text{span}(\vec{v}_1\cdots\vec{v}_r)\perp\text{Null}(A)$
- $\text{col}(V_{n-r})=\text{span}(\vec{v}_{r+1}\cdots\vec{v}_n)=\text{Null}(A)$

SVD ex (HW10 5d)

$A=\begin{bmatrix}-1&4\\1&4\end{bmatrix}$ Find the SVD.

① $A^TA=\begin{bmatrix}2&0\\0&32\end{bmatrix}$

② $\det(A^TA-\lambda I)=(2-\lambda)(32-\lambda)$

$\lambda_1=32,\ \lambda_2=2$

④ $\sigma_1=\sqrt{32}$ $\sigma_2=\sqrt{2}$

$=4\sqrt{2}$

③ $\text{Null}(A^TA-\lambda I)$

↳ $\vec{v}_1=\begin{bmatrix}0\\1\end{bmatrix}$ $v_2=\begin{bmatrix}1\\0\end{bmatrix}$

⑤ $\vec{u}_1=\frac{1}{\sigma_1}A\vec{v}_1$

$=\frac{1}{4\sqrt{2}}\begin{bmatrix}-1&4\\1&4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

$=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

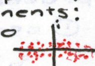
$\vec{u}_2=\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

$A=U\Sigma V^T$

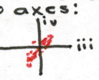
$=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}4\sqrt{2}&0\\0&\sqrt{2}\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$

PCA projections

- along principal components:
  - data aligned to orthogonal axes



  - axis with larger spread (i) corresponds to larger singular value (so ii would be $\vec{v}_1$, i would be $\vec{v}_2$)
- along random directions
  - not aligned to axes:



PCA

Algorithm

① Arrange data $(\vec{x}_1\cdots\vec{x}_n)$ into a matrix (either as columns or rows)

$$X=\begin{bmatrix}|&&|\\\vec{x}_1&\cdots&\vec{x}_n\\|&&|\end{bmatrix}\ \underline{\text{OR}}\ \begin{bmatrix}—\vec{x}_1—\\\vdots\\—\vec{x}_n—\end{bmatrix}$$

② SVD: $X=U\Sigma V^T=\sum\sigma_i\vec{u}_i\vec{v}_i^T$

③ Find 1st k principal components

↳ if data is cols, choose $\vec{u}_1\cdots\vec{u}_k$

↳ if data is rows, choose $\vec{v}_1\cdots\vec{v}_k$

④ Project data onto principal components to get lower-dim structure:

↳ cols: projection of $\vec{x}_i$ on the k-dim subspace has coeff. $U_k^T\vec{x}_i$.

and projection $U_kU_k^T\vec{x}_i=\sum_{p=1}^{k}(\vec{u}_p^T\vec{x}_i)\vec{u}_p$.

projection for all columns is $U_kU_k^TX$.

↳ rows:

coeff : $V_k^T\vec{x}_i$

proj (vector): $V_kV_k^T\vec{x}_i=\sum_{p=1}^{k}(\vec{v}_p^T\vec{x}_i)\vec{v}_p$

proj (all): $XV_kV_k^T$

PCA via Minimizing Reconstruction Error

→ want to minimize the squared reprojection error $\|\vec{x}_i-(\vec{x}_i^T\vec{w})\vec{w}\|^2$ for a data point $\vec{x}_i$, if we call the (normalized) direction $\vec{w}$ that we're projecting on.

want to solve:

$$\underset{\|\vec{w}\|=1}{\text{argmin}}\sum_{i=1}^{n}\|\vec{x}_i-(\vec{x}_i^T\vec{w})\vec{w}\|^2$$

derivation:

$\|\vec{x}_i-(\vec{x}_i^T\vec{w})\vec{w}\|^2=(\vec{x}_i-(\vec{x}_i^T\vec{w})\vec{w})^T(\vec{x}_i-(\vec{x}_i^T\vec{w})\vec{w})$

$=\vec{x}_i^T\vec{x}_i+(\vec{x}_i^T\vec{w})^2\vec{w}^T\vec{w}-2(\vec{x}_i\vec{w})\vec{x}_i^T\vec{w}$

$=\|\vec{x}_i\|^2+(\vec{x}_i^T\vec{w})^2-2(\vec{x}_i^T\vec{w})^2=\underset{\downarrow\text{constant}}{\|\vec{x}_i\|^2}-\underset{\text{want to minimize this}}{(\vec{x}_i^T\vec{w})^2}$

$\underset{\|\vec{w}\|=1}{\text{argmax}}\sum_{i=1}^{n}(\vec{x}_i^T\vec{w})^2=\underset{\|\vec{w}\|=1}{\text{argmax}}\sum_{i=1}^{n}\vec{w}^T\vec{x}_i\vec{x}_i^T\vec{w}$

$=\underset{\|\vec{w}\|=1}{\text{argmax}}\ \vec{w}^TXX^T\vec{w}=\underset{\|\vec{w}\|=1}{\text{argmax}}\ w^TU\Sigma V^TV\Sigma^TU^T\vec{w}$

$$\boxed{=\underset{\|\vec{w}\|=1}{\text{argmax}}\ \vec{w}^TU\Sigma\Sigma^TU^T\vec{w}}$$

↳ do a change of basis with $U^T\vec{w}=\vec{\tilde{w}}$

since orthonormal matrices don't affect norms, we still have $\|\vec{\tilde{w}}\|=1$

$\underset{\|\vec{\tilde{w}}\|=1}{\text{argmax}}\ \vec{\tilde{w}}^T\begin{bmatrix}\sigma_1^2&&\\&\ddots&\\&&\sigma_m^2\end{bmatrix}\vec{\tilde{w}}$

→ direction that maximizes the reconstruction error will be the 1st principal component $\vec{u}_1$

↳ principal components reduce orthogonal projection error

proof $N(A)=N(A^TA)$

① $N(A)\subseteq N(A^TA)$ $\vec{v}\in N(A)$ wts $\vec{v}\in N(A^TA)$

$A\vec{v}=\vec{0}\Rightarrow A^TA\vec{v}=A\vec{0}=\vec{0}$

$\vec{v}\in N(A^TA)$

② $N(A)\supseteq N(A^TA)$ $\vec{v}\in N(A^TA)$ wts $\vec{v}\in N(A)$

$A^TA\vec{v}=\vec{0}$

$\vec{v}^TA^TA\vec{v}=\vec{v}^T\vec{0}=0$

$\|A\vec{v}\|^2=0\rightarrow\|A\vec{v}\|=0\Rightarrow A\vec{v}=0$

Minimum Energy control

- solve $C\vec{w}=\vec{x}^*$ s.t. $\|\vec{w}\|$ is minimized. $(C=C_r)$

$C=U\Sigma V^T$ $A\vec{v}_i=\sigma_i\vec{u}_i$ for $i\leq r$

$A\vec{v}_i=\vec{0}$ for $i>r$

$C\vec{w}=\vec{x}^*$

$A(\sum_{i=1}^{r}\langle\vec{w},\vec{v}_i\rangle\vec{v}_i)=\sum_{i=1}^{r}\langle\vec{x}^*,\vec{u}_i\rangle\vec{u}_i$

$\sum_{i=1}^{r}\langle\vec{w},\vec{v}_i\rangle(A\vec{v}_i)=\sum_{i=1}^{r}\langle\vec{x}^*,\vec{u}_i\rangle\vec{u}_i$

$\sum_{i=1}^{r}\sigma_i\langle\vec{w},\vec{v}_i\rangle\vec{u}_i=\sum_{i=1}^{r}\langle\vec{x}^*,\vec{u}_i\rangle\vec{u}_i$

$\sigma_i\langle\vec{w}_i,\vec{v}_i\rangle=\langle\vec{x}^*,\vec{u}_i\rangle$

$\Rightarrow\langle\vec{w},\vec{v}_i\rangle=\frac{\langle\vec{x}^*,\vec{u}_i\rangle}{\sigma_i}$

For $\forall i\leq r$

$$\boxed{\vec{w}=\sum_{i=1}^{r}\frac{\langle\vec{x}^*,\vec{u}_i\rangle}{\sigma_i}\vec{v}_i}$$

Frobenius norm

$\|A\|_F^2=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}^2$ ← Frobenius norm of matrix A

$\|\vec{v}_{ij}\|^2=\sum_{i=1}^{m}A_{ij}^2$ ← squared norm of a particular vector

$\|A\|_F^2=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}^2=\sum_{j=1}^{n}(\sum_{i=1}^{m}A_{ij}^2)=\sum_{j=1}^{n}\|\vec{v}_j\|^2$

Frobenius Norms and orthonormal C. of Basis

$\|QM\|_F^2=\|\begin{bmatrix}|&&|\\Q\vec{v}_1&\cdots&Q\vec{v}_n\\|&&|\end{bmatrix}\|_F^2=\sum_{j=1}^{n}\|Q\vec{v}_j\|^2=\sum_{j=1}^{n}\vec{v}_j^TQ^TQ\vec{v}_j$

$=\sum_{j=1}^{n}\vec{v}_j^T\vec{v}_j=\sum_{j=1}^{n}\|\vec{v}_j\|^2=\|M\|_F^2$

Frob. Norm & SVD

$\|A\|_F=\|U\Sigma V^T\|=\|U^TU\Sigma V^TV\|_F=\|\Sigma\|_F=\sqrt{\sum_{i=1}^{r}\sigma_i^2}$

Classification

- define $\vec{w}$ s.t $\vec{w}^T\vec{x}>0$ (+) or $\vec{w}^T\vec{x}<0$ (−)

  so $\vec{\ell}=\vec{x}^T\vec{w}$ represents predicted class labels $\vec{\ell}$ for points $\vec{x}$

  → need to figure out what $\vec{w}$ is.
- cost fcns → help evaluate how good our weight ($\vec{w}$) is.
- three cost fcns:

  ① Least squares / squared loss: problems due to reliance on means

  $C_i(p)=(p-\ell_i)^2$

  ② Exponential loss: severly penalizes wrong guesses

  $C_i(p)=e^{-\ell_ip}$

  ③ Logistic loss: actually used in classification

  $\ln(1+e^{-p})$
- Algorithm for optimizing cost fcns:

  ① Arbitrarily choose operating point $\vec{w}_k=\vec{w}[0]=\vec{0}$

  ② Quadracize around $\vec{w}_k$

  $f(\vec{w})=\sum_{i=1}^{m}c(\vec{x}_i^T\vec{w},\ell_i)$ around $\vec{w}_k$

  $f(\vec{w})\approx\vec{w}^TA\vec{w}+\vec{b}^T\vec{w}+d$ (generic form of quadratic)

  ③ Find the minimizer of the quadratic. call this $\vec{w}[1]$

  ④ Set $\vec{w}_k=\vec{w}[1]$ and repeat from ②

Moore-Penrose (wide)

$A\vec{x}=\vec{y}$

$A^+A\vec{x}=A^+\vec{y}$

$V\Sigma^+U^TU\Sigma V^T\vec{x}=A^+\vec{y}$

$\vec{x}=V\Sigma^+\Sigma V^T\vec{x}$

$$\boxed{\hat{\vec{x}}=V\Sigma^+U^T\vec{y}}$$

$\Sigma^+=\begin{bmatrix}\frac{1}{\sigma_0}&&&\\&\frac{1}{\sigma_1}&&\\&&\ddots&\frac{1}{\sigma_m}\\0&\cdots&&0\end{bmatrix}$

Quadratic Approximation

$$f(\vec{x})\approx f(\vec{x}^*)+\frac{\partial f}{\partial\vec{x}}(\vec{x}^*)(\vec{x}-\vec{x}^*)+\frac{1}{2}(\vec{x}-\vec{x}^*)^T\left(\frac{\partial^2f}{\partial\vec{x}^2}(\vec{x}^*)\right)(\vec{x}-\vec{x}^*)$$

Hessian of f $((H_{\vec{x}}f)$ or $(\frac{\partial^2f}{\partial x})$

$H_{\vec{x}}f$ $\frac{\partial^2f}{\partial\vec{x}^2}=\begin{bmatrix}\frac{\partial^2f}{\partial x_1^2}&\cdots&\frac{\partial^2f}{\partial x_1\partial x_n}\\\vdots&&\vdots\\\partial^2f&\cdots&\partial^2f\end{bmatrix}$

Moore-penrose pseudoinverse

returns min-norm, least squares solution

$\underset{\vec{x}}{\text{argmin}}\|\vec{x}\|^2$ st $\|A\vec{x}-\vec{y}\|^2=\underset{\vec{s}}{\min}\|A\vec{s}-\vec{y}\|$

Min energy control

$\vec{x}[n]=A^n\vec{x}[0]+C_n\vec{u}$

$C_n\vec{u}=\vec{x}[n]-A^n\vec{x}[0]$

$\vec{u}=C_n^{-1}(\vec{x}[n]-A^n\vec{x}[0]$

↳ exactly n timesteps

$\vec{x}[t]=A^t\vec{x}[0]+C_t\vec{u}$

$\hat{\vec{u}}=C_t^+(\vec{x}[t]-A^t\vec{x}[0]$

$\vec{u}=\text{argmin}\|\vec{u}\|$

↳ use controllability matrix pseudoinverse to get min energy control vector min energy norm → underdetermined/wide

$z = (A^TA)^{-1}A^T\vec{x}$

- orthogonality:
  $<\vec{v},\vec{w}> = \vec{v}^T\vec{w} = \vec{w}^T\vec{v} = 0$
  ↳ $\vec{v} \perp \vec{w}$

- orthogonal projection of $\vec{y}$ onto $\vec{b}$:

$$\vec{y}_{\vec{b}} = \frac{\vec{y}^T\vec{b}}{\|\vec{b}\|^2}\vec{b}$$

- least squares
  - orthogonal proj. of $\vec{y}$ onto subspace spann
    by col(A)

$$\vec{y}_A = A\hat{\underline{x}} = \underline{A(A^TA)^{-1}A^T\vec{y}}$$

if col of A are mutually orthogona

Projection onto an orthogonal matrix

$$\vec{y}_{A_n} = \left(\frac{\vec{v}_1^T\vec{y}}{\|\vec{v}_1\|^2}\right)\vec{v}_1 + \cdots + \left(\frac{\vec{v}_n^T\vec{y}}{\|\vec{v}_n\|^2}\right)\vec{v}_n$$

Projection onto orthonormal matrix

$$\vec{y}_{A_n} = (\vec{v}_1^T\vec{y})\vec{v}_1 + \cdots + (\vec{v}_n^T\vec{y})\vec{v}_n$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} — \vec{v}_1 — \\ \vdots \\ — \vec{v}_n — \end{bmatrix}\vec{y} = A_nA_n^T\vec{y}$$

the proj. of $\vec{y}$ onto span(A) is sum of proj of $\vec{y}$ on each of the individual columns of A

$$A_2 = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \quad \vec{y}_{A_2} = \vec{y}_{\vec{v}_1} + \vec{y}_{\vec{v}_2}$$

Gram-Schmidt

recursive def:

$$\vec{q}_k = \frac{\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k)}{\|\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k)\|}$$

procedure:

① compute:

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

② $\vec{e}_i = \vec{v}_i - \sum_{\ell=1}^{i-1}(q_\ell^T\vec{v}_i)\vec{q}_\ell$

$$\vec{e}_2 = \vec{v}_2 - \frac{<\vec{v}_2,\vec{q}_1>}{\|\vec{q}_1\|^2}\vec{q}_1 = \vec{v}_2 - <\vec{v}_2,\vec{q}_1>\vec{q}_1$$

$$\vec{e}_3 = \vec{v}_3 - (<\vec{v}_3,\vec{q}_1>\vec{q}_1 + <\vec{v}_3,\vec{q}_2>\vec{q}_2)$$

③ $\vec{q}_i = \frac{\vec{e}_i}{\|\vec{e}_i\|}$

$$\vec{q}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|} \qquad \vec{q}_3 = \frac{\vec{e}_3}{\|\vec{e}_3\|}$$

Proof of Orthonormality

① Show each vector is normal (true by construction)

for $\forall\vec{v}_i$: $\left\|\frac{\vec{v}_i}{\|\vec{v}_i\|}\right\| = \frac{1}{\|\vec{v}_i\|}\|\vec{v}_i\| = 1$

② orthogonality
(induction → k-1 vectors orthonormal)
↳ wts $\vec{q}_k$ orthogonal to prev. ($\vec{q}_p^T\vec{q}_k = 0$) for $\forall p = 1,\ldots,k-1$
↳ def $A = \frac{1}{\text{norm factor in } \vec{q}_k}$

$$\vec{q}_p^T\vec{q}_k = A\vec{q}_p^T(\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k))$$

$$= A(\vec{q}_p^T\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_p^T\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k))$$

$$= A(\vec{q}_p^T\vec{v}_k - \vec{q}_p^T\vec{q}_p(\vec{q}_p^T\vec{v}_k))$$

$$= A(\vec{q}_p^T\vec{v}_k - \vec{q}_p^T\vec{v}_k)$$

$$= 0$$

since (k-1) vectors are orthonormal, $\vec{q}_p^T\vec{q}_\ell$ is only nonzero when $\ell = p$.

Proof of equivalent span

① Assume k-1 vectors of V & Q span same space.
② wts $\vec{v}_k$ written as linear combo of $\{\vec{q}_1\cdots\vec{q}_k\}$ (by construction of $\vec{q}_k$):

$$\vec{v}_k = \left\|\vec{v}_k - \sum_{\ell=1}^{k-1}(\vec{q}_\ell^T\vec{v}_k)\vec{q}_\ell\right\|\vec{q}_k + \sum_{\ell=1}^{k-1}(\vec{q}_\ell^T\vec{v}_k)\vec{q}_\ell$$

↳ proves span($\{\vec{v}_1\cdots\vec{v}_k\}$) ⊆ span($\{\vec{q}_1\cdots\vec{q}_k\}$)

span($\{\vec{v}_1\cdots\vec{v}_k\}$) ⊇ span($\{\vec{q}_1\cdots\vec{q}_k\}$)

$$\vec{q}_k = \frac{\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k)}{\|\vec{v}_k - \sum_{\ell=1}^{k-1}\vec{q}_\ell(\vec{q}_\ell^T\vec{v}_k)\|}$$

QR Decomposition

- gives a way to transform btwn G-S ⊥ ‖·‖-ized $\vec{q}_i$ and og vectors $\vec{v}_i$ through upper triangularized matrix R.
- any matrix V can be decomposed into matrix w/orthonormal cols (Q) and upper triang. matrix (R)

$$\vec{v}_1 = \|\vec{e}_1\|\vec{q}_1$$

$$\vec{v}_2 = \|\vec{e}_2\|\vec{q}_2 + (\vec{q}_1^T\vec{v}_2)\vec{q}_1$$

$$\vec{v}_n = \|\vec{e}_n\|\vec{q}_n + \sum_{\ell=1}^{n-1}(\vec{q}_\ell^T\vec{v}_n)\vec{q}_\ell$$

$$[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n] = [\vec{q}_1\ \vec{q}_2 \cdots \vec{q}_n]\begin{bmatrix} \|\vec{e}_1\| & \vec{q}_1^T\vec{v}_2 & \vec{q}_1^T\vec{v}_3 \cdots \vec{q}_1^T \\ 0 & \|\vec{e}_2\| & \vec{q}_2^T\vec{v}_3 \cdots \vec{q}_2^T \\ 0 & 0 & \|\vec{e}_3\| \cdots \vec{q}_3^T \\ \vdots & \vdots & \ddots \\ 0 & 0 & 0 \cdots \|\vec{e} \end{bmatrix}$$

$$V = QR$$

More spectral Theorem

- for real, symmet $\underset{n\times n}{A}$, $A = U\Lambda U^T$

mini-proof sketch:

$A = UTU^T$

$A^T = UT^TU^T$

$T = \Lambda$

$A = U\Lambda U^T$

- for symmetric matrices EVD & SVD are same upto (+)s & (– in eigenvectors & reordering eigenva

UPPER TRIANGULAR

- done for defective (non-diagnolizable) matrices
- convert $\underset{n\times n}{A}$ into form T, using change of basis via U.

$A = UTU^{-1}$
↳ T has λ_i's on the diagonals

- must have at least 1 eigenvector to upper-triang.

$$A = UTU^{-1}$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & ? & ? & ? \\ 0 & \ddots & & ? \\ \vdots & & \ddots & ? \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

Upper Triangularization Procedure

(3×3 case)

$A = U_3TU_3$

$U_3^TAU_3 = T$

$U_3 = [\vec{v}_1\ \ R_2U_2]$

first eigenvector of A

extension of $\vec{v}_1$ using Gram-Schmidt

$$U_3^TAU_3 = \begin{bmatrix} \lambda_1 & \vec{?} \\ \vec{0} & U_2^T\underbrace{R_2^TAR_2}_{Q}U_2 \end{bmatrix}$$

$Q = R_2^TAR_2$ ← define Q. solve this out.

$U_2 = [\vec{v}_2\ \ \vec{v}_3]$ ← eigenvector(s) of Q
↳ if 1 e-vec, do G-S

$U_3 = [\vec{x}\ \ \vec{y}\ \ \vec{z}] = [\vec{v}_1\ \ R_2U_2]$ ← Plug U_2 back into previously defined U_3

$= [\vec{v}_1\ \ R_2\vec{v}_2\ \ R_2\vec{v}_3]$

$U_3^TAU_3 = T$ ← plug U_3 into previously defined T, and you have your upper triangular matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$$

$A\vec{v}_1 = \lambda_1\vec{v}_1$

$$\lambda_1 = 2,\ \vec{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

→ $U_3^TAU_3 = T$

$$T = \begin{bmatrix} 2 & 0 & 4\sqrt{3} \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

↳ do gram-schmidt with $\vec{v}_1$, get R_2

$$\begin{bmatrix} \sqrt{2}/2 & 1 & 0 & 0 \\ \sqrt{2}/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$ GS

$$R_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$U_3 = [\vec{v}_1\ \ R_2U_2]$

$Q = R_2^TAR_2$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -2 \end{bmatrix}$$

↳ find eigenvector(s) of Q
if not 2 e-vcs → do GS.
these form U_2

$$U_2 = [v_2\ v_3] = \begin{bmatrix} -\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

↳ $\lambda_2 = 2$ ↳ $\lambda_3 = -4$

$U_3 = [\vec{v}_1\ \ R_2\vec{v}_2\ \ R_2\vec{v}_3]$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

"Alternate" view

$U_n = [\vec{v}_1\ \ R_{n-1}U_{n-1}] = [\vec{v}_1\ \ R_1\vec{v}_2\ \ R_1R_2\vec{v}_3\cdots]$

↳ find possible eigenvalues/vectors (at least 1), extend $\mathbb{R}^n$.

$$U = [\vec{v}_1\ \ R] \quad U^TAU = \begin{bmatrix} \vec{v}_1^TA\vec{v}_1 & \vec{v}_1^TAR \\ R^TA\vec{v}_1 & R^TAR \end{bmatrix}$$

→ recurse on R^TAR

$$= \begin{bmatrix} \lambda_1 & \vec{v}_1^TAR \\ \vec{0} & R^TAR \end{bmatrix}$$

$R^TAR = U_2T_2U_2^T$ → $T_2 = U_2^TR^TT_2RU_2$

$T_2 = (RU_2)^T\,T_2(RU_2)$ → $[\vec{v}_1\ \ RU_2]$

$$\begin{bmatrix} \vec{v}_1A\vec{v}_1 & \vec{v}_1^TARU_2 \\ \vec{0} & U_2^TR^TARU_2 \end{bmatrix}$$

Spectral Theorem

- real, symmetric n×n matrices can always be diagnolized

① all eigenvalues are real-valued
② there is a full set of n orthogonal and LI real eigenvectors

↳ used heavily in SVD ↳ A^TA, AA^T are both symmetric matrices and can be diagonalized with orthonormal eigenvectors which is why U, V in SVD are orthonormal

Linearization

linearity:
- system is linear if it follows scaling & additivity properties:

① scaling: $f(ax) = af(x)$

② additivity: $f(x+y) = f(x) + f(y)$

linearizing a nonlinea system using First-order Taylor approximation

$$f(x) \approx f(x^*) + \frac{df}{dx}\bigg|_{x=x^*}(x-x^*)$$

$$f(x,u) \approx f(x^*,u^*) + \frac{df}{dx}\bigg|_{x=x^*}(x-x^*) + \frac{df}{du}\bigg|_{u=u^*}(u-u^*)$$

Taylor Series

$$f(x) \approx f(x^*) + f'(x^*)(x-x^*) + \frac{f''(x^*)}{2!}(x-x^*)^2 + \cdots + \frac{f^{(n)}(x^*)}{n!}(x-$$

Linearizing a System of Nonlinear Equations

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_m(x_1,\ldots,x_n) \end{bmatrix}$$

system of nonlinear functions

- can linearize each equation using the partial derivative of the fcn wrt each state variable:

$$f_1(\vec{x}) \approx f_1(\vec{x}^*) + \frac{\partial f_1}{\partial x_1}\bigg|_{x_1=x_1^*}(x_1-x_1^*) + \cdots + \frac{\partial f_1}{\partial x_n}\bigg|_{x_n=x_n^*}(x_n-x_n^*)$$

↳ eg for case f_1. have to do this for $\forall f_1\cdots f_n$

- can represent as a matrix-vector equation (the Jacobian matrix of $\vec{f}$ wrt $\vec{x}$). Note that $\vec{\delta x}$ is the distance for the equilibrium point ($\vec{\delta x} = \vec{x} - \vec{x}^*$)

Jacobian

$$\vec{f}(\vec{x}) = J_{\vec{x}}\vec{\delta x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{x_1^*} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{x_n^*} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_{x_1^*} & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_{x_n^*} \end{bmatrix}\begin{bmatrix} (x_1-x_1^*) \\ \vdots \\ (x_n-x_n^*) \end{bmatrix}$$

Jacobian with vector input

$$\vec{f}(\vec{x},\vec{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{x_1^*} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{x_n^*} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_{x_1^*} & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_{x_n^*} \end{bmatrix}\begin{bmatrix} (x_1-x_1^*) \\ \vdots \\ (x_n-x_n^*) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1}\Big|_{u_1^*} & \cdots & \frac{\partial f_1}{\partial u_k}\Big|_{u_k^*} \\ \vdots & & \\ \frac{\partial f_m}{\partial u_1}\Big|_{u_1^*} & & \frac{\partial f_m}{\partial u_k}\Big|_{u_k^*} \end{bmatrix}\begin{bmatrix} (u_1-u_1^*) \\ \vdots \\ (u_k-u_k^*) \end{bmatrix} = J_{\vec{x}}\delta_x + J_{\vec{u}}$$

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x},\vec{u}) = \vec{f}(\vec{x}^*,u^*) + \frac{\partial\vec{f}}{\partial\vec{x}}(\vec{x}^*,u^*)(\vec{x}-\vec{x}^*) + \frac{\partial\vec{f}}{\partial\vec{u}}(\vec{x}^*,\vec{u}^*)(u-u^*) + \vec{\omega}$$

- equilibrium points: when $\vec{f}(\vec{x}^*,u^*) = \vec{0}$
- stability of system → check e-vals of $J_{\vec{x}}$

can use feedback is controllable

$(J_{\vec{x}} + J_{\vec{u}}K)$ → actual control applied will be

Transistors

PMOS

$V_{GS} \leq -|V_{tp}| \rightarrow$ closed switch

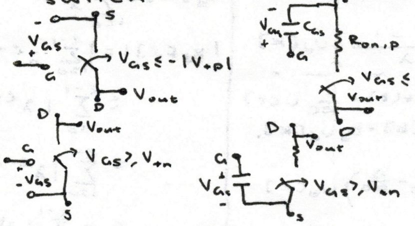
- closed switch when V_G is at least $|V_{tp}|$ below V_S (low gate voltage), otherwise open.

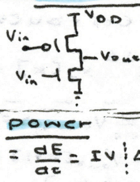
NMOS

$V_{GS} \geq |V_{tn}| \rightarrow$ closed switch

- closed switch when V_G is at least V_{tn} above source voltage V_S (gate voltage high), otherwise closed
- HIGH INPUT → CLOSED

voltage-cont. switch





$V_{in} = V_{DD} \rightarrow V_{out} = 0$

$V_{in} = 0 \rightarrow V_{out} = V_{DD}$

$I(t) = C\frac{dV_c(t)}{dt}$

inductor

$V_L(t) = L\frac{d}{dt}I_L(t)$

power

$P = \frac{dE}{dt} = IV$ | $\Delta E \int_A^B P(t)dt$

$P = I^2R = \frac{V^2}{R}$

Energy

$\Delta E_{charge} = \frac{1}{2}CV_s^2$

$\Delta E_{discharging} = -\frac{1}{2}CV_s^2$

Differential Equations

Homogeneous:

$\frac{d}{dt}x(t) = \lambda x(t)$

$x(t) = x(0)e^{\lambda t}$

Nonhomogeneous:

$\frac{d}{dt}x(t) = \lambda x(t) + \alpha$

$x(t) = x(0)e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$

Non-homog. w/ u(t)

$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$

$x(t) = x(0)e^{\lambda t} + \int_0^t bu(\tau)e^{\lambda(t-\tau)}d\tau$

Diagonalization

$\vec{y} = A\vec{x}$ $V = [\vec{v}_1 \ldots \vec{v}_n]$ $\Lambda = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$

$\vec{x} = V\tilde{\vec{x}}$ $\tilde{\vec{x}} = V^{-1}\vec{x}$ $\vec{y} = V\tilde{\vec{y}}$

$V\tilde{\vec{y}} = AV\tilde{\vec{x}}$ | $\Lambda = V^{-1}AV$

$\tilde{\vec{y}} = V^{-1}AV\tilde{\vec{x}}$ | $A = V\Lambda V^{-1}$

$\tilde{\vec{y}} = \Lambda\tilde{\vec{x}}$

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \vec{y} \arrow[d, bend left] \\
\tilde{\vec{x}} \arrow[u, bend right, "V"'] \arrow[r, "\Lambda"'] & \tilde{\vec{y}} \arrow[u, bend left, "V"']
\end{tikzcd}$$

V^{-1}

Multivar. Diff Eq

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$

→ Transform to e-basis coordinates:

$\frac{d}{dt}\tilde{\vec{x}}(t) = V^{-1}AV\tilde{\vec{x}}(t)$

$= V^{-1}A\vec{x}(t)$

$= \Lambda\tilde{\vec{x}}(t)$

→ solve $\tilde{\vec{x}}(0)$

$\tilde{\vec{x}}(0) = V^{-1}x(0)$

→ plug in for $\frac{d}{dt}\tilde{\vec{x}}(t)$'s and solve for $\tilde{\vec{x}}(t)$

→ back to OG coords

$\vec{x}(t) = V\tilde{\vec{x}}(t)$

Eigenvals in Time Domain

- real ($\lambda = a$)
  → decaying exponential, no oscillations
- imaginary ($\lambda = bj$)
  → oscillating sinusoid that doesn't decay
- complex ($\lambda = a+bj$)
  → oscillates as it decays

complex numbers

$z = x + jy = |z|e^{j\theta}$ ← form

$|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$ ← magnitude

$\theta = atan2(y, x)$ ← phase

$x = |z|\cos\theta$ $y = |z|\sin\theta$

$e^{j\theta} = \cos\theta + j\sin\theta$ ← Euler

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ | $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

complex conjugates

$\bar{z} = x - jy = |z|e^{-j\theta}$ ← form

$\overline{(z+w)} = \bar{z} + \bar{w}$ ← add/subtract

$\overline{(zw)} = \bar{z}\bar{w}$ ← mult/divide

$\overline{(z^n)} = \bar{z}^n$ ← power

$\bar{z} = z \Rightarrow z$ is real

$\bar{z} = -z \Rightarrow z$ is purely complex

Complex algebra

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ | $z_1z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$

$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$

$z^n = |z|^n e^{jn\theta}$

$z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}}e^{j\frac{\theta}{2}}$

Frequency Domain Phasors & impedence

$V(t) = V_0\cos(\omega t + \phi_v) = Re(V_0e^{j\phi_v}e^{j\omega t})$ $\tilde{V} = V_0e^{j\phi_v}/2$

$I(t) = I_0\cos(\omega t + \phi_i) = Re(I_0e^{j\phi_i}e^{j\omega t})$ $\tilde{I} = I_0e^{j\phi_i}/2$

mepedence (Z): $Z = \frac{\tilde{V}}{\tilde{I}}$

$Z_R = R$ $Z_C = \frac{1}{j\omega C}$ $Z_L = j\omega L$

Transfer functions

$H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$

$H(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)}$

Relations

$-1 = j^2 = e^{j\pi} = e^{-j\pi}$

$\sqrt{-1} = j = e^{j\frac{\pi}{2}}$

$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$

$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\pi/4} = \frac{\pm(1+j)}{\sqrt{2}}$

Rotation matrix

$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

sin & cos

$\sin(-\theta) = -\sin(\theta)$

$\sin^2\theta + \cos^2\theta = 1$

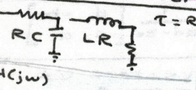
$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin\theta = \cos(\frac{\pi}{2} - \theta)$ $\cos\theta = \sin(\frac{\pi}{3} - \theta)$

Low-pass filters



$\tau = RC$

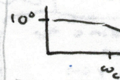
$H(j\omega) = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega L/R}$

$= \frac{1}{1+j\omega/\omega_c}$

$\omega < \omega_c \rightarrow |H(j\omega)| = 1$

$\omega > \omega_c \rightarrow |H(j\omega)| = \frac{\omega_c}{\omega}$

$\omega_c = \omega \downarrow \frac{1}{\sqrt{2}}$



$|H(j\omega)|$ vs ω

↑ magnitude

phase

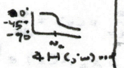
$\omega < \omega_c/10 \rightarrow \theta = 0°$

$\omega = 10\omega_c \rightarrow \theta = -84.3°$

$\omega = \omega_c/10 \rightarrow \theta = -5.7°$

$\omega = \omega_c = -45°$

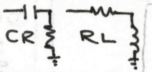
$\omega > 10\omega_c \rightarrow \theta = 90°$



cutoff frequency $(\omega_c) = \frac{1}{RC}$

High pass filters

$\tau = L/R$



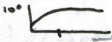
$\omega < \omega_c \rightarrow |H(j\omega)| = \frac{\omega}{\omega_c}$

$\omega > \omega_c \rightarrow |H(j\omega)| = 1$

$\omega = \omega_c \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}}$

$H(j\omega) = \frac{j\omega RC}{1+j\omega RC} = \frac{j\omega L/R}{1+j\omega L/R}$

$= \frac{j\omega/\omega_c}{1+j\omega/\omega_c}$



cutoff freq.

$\frac{R}{L} = \omega_c = \frac{1}{RC}$

$\omega < \omega_c/10 \rightarrow \theta = 90 = \pi/2$

$\omega = \omega_c/10 \rightarrow \theta = 84.3$

$\omega = \omega_c \rightarrow \theta = 45$

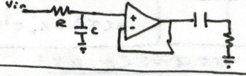
$\omega = 10\omega_c \rightarrow \theta = 5.7$

$\omega > 10\omega_c \rightarrow \theta = 0$

Bandpass filters

- HP & LP w/ buffer between

$H(j\omega) = \frac{j\omega RC}{1+j\omega RC} \cdot \frac{1}{1+j\omega RC}$

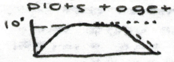


← example, obviously not the only BPF that exists

Higher-order Filters

eg: $H(\omega) = \frac{j\omega/\omega_{c1}}{1+j\omega/\omega_{c1}} \cdot \frac{1}{1+j\omega/\omega_{c2}}$

- magnitude: mult plots together



- Phase: add, then plot together

Bode plot note: If the filter has some gain, the maximum val of the transfer function ≠1

eg. if $|H(j\omega)| = K\frac{1}{1+j\omega/\omega_c}$, $|H(j\omega)|$ at ω_c will be $K/\sqrt{2}$ not $1/\sqrt{2}$

Stability (BIBO)

- discrete time: $\vec{x}[k+1] = A\vec{x}[k] + \vec{b}u[k]$
  - stable: $|\lambda| < 1$
  - unstable: $|\lambda| \geq 1$
- continuous: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$
  - stable: $Re(\lambda) < 0$
  - unstable: $Re(\lambda) > 0$
    - ↳ marginally unstable: $Re(\lambda) = 0$

controllability

- $\vec{x}[k+1] = A\vec{x}[k] + b\vec{u}[k]$ is controllable if, given any $\vec{x}[0]$, we can specify any inputs $u[0] \cdots u[n]$ to reach any state $\vec{x}[n]$
- check by making sure the controllability matrix (C) is full rank

$C = [\vec{b} \; A\vec{b} \ldots A^{n-1}\vec{b}]$

- if we know our system is controllable, can find $\vec{u}[n]$ to reach $\vec{x}[n]$ at a certain timestep:

$\vec{x}[n] = C\vec{u}[n]$

$\vec{u}[n] = C^{-1}\vec{x}[n]$

- feedback control with unstable eigenvalues: $\vec{x}[k+1] = A\vec{x}[k] + \underbrace{\vec{b}\vec{f}^T\vec{x}[k]}_{=u[k]}$ → $\vec{x}[k+1] = \underbrace{(A + \vec{b}\vec{f}^T)}_{\text{new state transisition matrix}}\vec{x}[k]$

→ can choose f_i values to make the system reach wherever we want

System identification

$$\begin{bmatrix}x_1[k+1] \\ \vdots \\ x_n[k+1]\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{in} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}x_1[k] \\ \vdots \\ x_n[k]\end{bmatrix} + \begin{bmatrix}b_1 \\ \vdots \\ b_n\end{bmatrix}u[k]$$

$\vec{p}$: vector of unknowns, populate with elements of A & $\vec{b}$

$\vec{p} = [a_{11} \ldots a_{in} \; b_{11} \ldots b_{in}]^T$

$\vec{y}$: output of the system, populate with states

$\vec{y} = [x_1[1] \cdots x_n[1] \; x_1[2] \ldots x_n[2] \cdots x_1[m] \cdots x_n[m]]^T$

$\vec{p} = (D^TD)^{-1}D^T\vec{y}$

$x_1[k+1] = a_{11}x_1[k] + a_{12}x_2[k] + b_1u[k]$

$x_2[k+1] = a_{21}x_1[k] + a_{22}x_2[k] + b_2u[k]$

} ↳ form D matrix of Data points (w/ LI col)

$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \vec{\omega}[t]$ $A\vec{x} \approx \vec{b}$ → $DP \approx S$

$$\underbrace{\begin{bmatrix}x[1] \\ \vdots \\ x[m]\end{bmatrix}}_{S} = \underbrace{\begin{bmatrix}x[0] & u_1[0] & u_2[0] \\ x[1] & u_1[1] & u_2[1] \\ \vdots & \vdots & \vdots \\ x[m-1] & u_1[m-1] & u_2[m-1]\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}a \\ b_1 \\ b_2\end{bmatrix}}_{\vec{p}}$$

scalar-ish case

$x[t+1] = ax[t] + b_1u[t] + w[t]$

$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + w(t)$

$$\underbrace{\begin{bmatrix}x_1[1] & x_2[1] \\ \vdots & \vdots \\ x_1[m] & x_2[m]\end{bmatrix}}_{S} = \begin{bmatrix}x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[m-1] & x_2[m-1] & u[m-1]\end{bmatrix}\begin{bmatrix}a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2\end{bmatrix}$$

atan2(b, a)

b	a	θ
	0	0
	0	π/2
	0	3π/2
	-1	π
	1	0

Transfer function plots & Vout

$V_{out}(t) = \overset{0.01}{|H(j\omega_s)|}\overset{10}{|V_{in}(t)|}\overset{\cos(10^4t + \frac{\pi}{3} - \pi)}{\cos(\omega_st + \phi + \measuredangle H(j\omega))}$

eg: $V_{in}(t) = 10\cos(\omega_st + \frac{\pi}{3})$ $\omega_s = 10^4$

↳ look for value of $H(j10^4)$ → get $|H(j\omega_s)|$ and $\measuredangle H(j\omega_s)$ from those. Plug in.

"2nd order" Diff. eqns

$\frac{d^2}{dt^2}x(t) = -\frac{k_1}{m}x(t) - \frac{k_2}{m}\frac{dx(t)}{dt} + \frac{1}{m}u(t)$

state: $\vec{y}(t) = \begin{bmatrix}x(t) \\ dx(t)/dt\end{bmatrix}$

$$\underbrace{\begin{bmatrix}dx(t)/dt \\ d^2x(t)/dt^2\end{bmatrix}}_{\frac{d\vec{y}(t)}{dt}} = \underbrace{\begin{bmatrix}0 & 1 \\ -\frac{k_1}{m} & -\frac{k_2}{m}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x(t) \\ \frac{d}{dt}x(t)\end{bmatrix}}_{\vec{y}(t)} + \underbrace{\begin{bmatrix}0 \\ \frac{1}{m}\end{bmatrix}}_{\vec{b}}u(t)$$

$\frac{d\tilde{\vec{y}}}{dt}(t) = \Lambda\tilde{\vec{y}}(t) + V^{-1}\vec{b}u(t)$

→ solve decoupled diff eqns

$\frac{d}{dt}\tilde{y}_1(t) = \tilde{y}_1(t) + u_0$

$\frac{d}{dt}\tilde{y}_2(t) = \tilde{y}_2(t) + u_0$

↳ solve for $\tilde{y}_1(t) = \ldots$

$\tilde{y}_2(t) = \ldots$

↳ solve for initial condition

$\tilde{\vec{y}}(0) = V^{-1}\vec{y}(0)$

→ plug into $\tilde{\vec{y}}(t)$ using initial co

→ transform to original basis

$\vec{y} = V\tilde{\vec{y}}$

a||b||c

$= \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$

$= \frac{abc}{ab+ac+bc}$

a||b

$= \frac{1}{\frac{1}{a} + \frac{1}{b}}$

$= \frac{ab}{a+b}$

Voltage Divider

$V_{out} = \frac{Z_2}{Z_1 + Z_2}V_{in}$

Matrix Multiplication

$\begin{bmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}$

$= \begin{bmatrix}a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}\end{bmatrix}$

hoosing ω_c

$\omega_c = \sqrt{\omega_{signal}\,\omega_{noise}}$

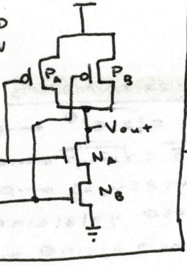
NAND Logic Gates

$V_A = V_B = V_{DD}$ ↳ $V_{out} = 0V$

$A = V_B = 0V$ ↳ $V_{out} = V_{DD}$

$=0,\ V_B = V_{DD}$ ↳ $V_{out} = V_{DD}$

$= V_{DD}, V_B = 0$ ↳ $V_{out} = V_{DD}$



$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$$

$$x_d[k+1] = e^{\lambda\Delta} x_d[k] + b\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right)u_d[k]$$

convert $\frac{d}{dt}V_{out}(t) = -\frac{1}{RC}V_{out}(t) + \frac{1}{RC}u(t)$ in form of $V_{out}[k+1] = \lambda_d V_{out}[k] + b_d u_d[k]$. What is λ_d & b_d?

$$V_{out}[k+1] = e^{-\frac{\Delta}{RC}} V_{out}[k] + \left(1 - e^{-\frac{\Delta}{RC}}\right)u_d[k]$$

$$\lambda_d = e^{-\frac{\Delta}{RC}} \qquad b_d = 1 - e^{-\frac{\Delta}{RC}}$$

Bounding: $Re\{\lambda\} < 0$

$$|x(t)| = \left|x(0)e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)}\omega(\tau)d\tau\right|$$

$$\leq |x(0)e^{\lambda t}| + \left|\int_0^t e^{\lambda(t-\tau)}\omega(\tau)d\tau\right|$$

$$\left|\int_0^t e^{\lambda(t-\tau)}\omega(\tau)d\tau\right| \leq \int_0^t |e^{\lambda(t-\tau)}\omega(\tau)|d\tau$$

$$= \int_0^t |e^{\lambda(t-\tau)}||\omega(\tau)|d\tau$$

$$= \int_0^t e^{\lambda_r(t-\tau)}|\omega(\tau)|d\tau$$

$$\leq \epsilon\int_0^t e^{\lambda_r(t-\tau)}d\tau$$

$$= \epsilon e^{\lambda_r t}\int_0^t e^{-\lambda_r \tau}d\tau$$

$$= \epsilon e^{\lambda_r t}\left[-\frac{1}{\lambda_r}e^{-\lambda_r\tau}\right]_0^t$$

$$= -\frac{\epsilon}{\lambda_r}e^{\lambda_r t}\left[e^{-\lambda_r \tau}\right]_0^t$$

$$= \frac{\epsilon e^{\lambda_r t}}{\lambda_r}\left(1 - e^{-\lambda_r t}\right)$$

$$= \frac{\epsilon}{\lambda_r}\left(e^{\lambda_r t} - 1\right)$$

↳ $\lambda_r < 0$ so $e^{\lambda_r t}$ bounded for all τ

⇒ system is stable

Geometric series:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

ounding? $Re\{\lambda\} = 0$

$$(t) = \int_0^t e^{\lambda(t-\tau)}\omega(\tau)d\tau$$

$$= \int_0^t e^{\lambda(t-\tau)}\epsilon e^{j\lambda_j\tau}d\tau$$

$$= \epsilon e^{j\lambda_j t}\int_0^t e^{-\lambda_j + \lambda_j\tau}d\tau$$

$$= \epsilon e^{j\lambda_j t}\int_0^t 1 d\tau$$

$$= \epsilon t e^{j\lambda_j t}$$

$$x_c(t)| = \epsilon t \Rightarrow \underset{\text{unstable}}{\infty}$$

$$x_d[\ell] = \sum_{i=0}^{\ell-1}\lambda^{\ell-i-1}\omega[i]$$

$$|x_d[\ell]| = \left|\sum_{i=0}^{\ell-1}\lambda^{\ell-i-1}\omega[i]\right|$$

$$\leq \sum_{i=0}^{\ell-1}|\lambda^{\ell-i-1}\omega[i]|$$

$$= \sum_{i=0}^{\ell-1}|\lambda^{\ell-i-1}||\omega[i]|$$

$$\leq \epsilon\sum_{i=0}^{\ell-1}|\lambda|^{\ell-i-1}$$

→ geometric series to get final upper bound:

$$|x_d[\ell]| \leq \epsilon\frac{1}{1-|\lambda|}$$

⇒ bounded even as $\ell \to \infty$

$|\lambda| < 1$

$$x_d[\ell] = \sum_{i=0}^{\ell-1}\lambda^{\ell-i-1}\omega[i]$$

$$= \epsilon\frac{\lambda^\ell - 1}{\lambda - 1}$$

$$|x_d[\ell]| = \epsilon\frac{|\lambda^\ell - 1|}{\lambda - 1}$$

⇒ goes to ∞ as $\ell \to \infty$

$\lambda > 1$

$$x_d[\ell] = \sum_{i=0}^{\ell-1}\lambda^{\ell-i-1}\omega[i]$$

$$= \epsilon\sum_{i=0}^{\ell-1}\lambda^{\ell-1}$$

$$=$$

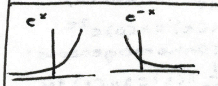
$$= \epsilon\ell\lambda^{\ell-1}$$

$$\rightarrow |x_d[\ell]| = \ell|\epsilon|$$

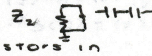
unbounded bc system

$|\lambda| = 1$

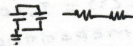
T-tera	10^{12}
G-giga	10^9
M-mega	10^6
k-kilo	10^3
h-hecto	10^2
da-deca	10^1
d-deci	10^{-1}
c-centi	10^{-2}
m-milli	10^{-3}
M-micro	10^{-6}
n-nano	10^{-9}
p-pico	10^{-12}
f-femto	10^{-15}



- Resistors in parallel & capac in series are combined using $Z_1 \parallel Z_2$



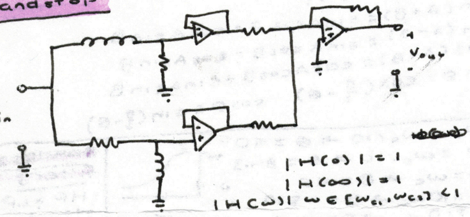
- Resistors in series & capacitor in parallel are combined using $Z_1 + Z_2$



Matrix inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and stop



$|H(0)| = 1$

$|H(\infty)| = 1$

$|H(\omega)|\ \omega \in [\omega_{c1}, \omega_{c2}] < 1$

$$H(\omega) = \left(\frac{1}{1 + j\omega/\omega_{c1}} + \frac{j\omega/\omega_{c2}}{1 + j\omega/\omega_{c2}}\right)$$

Using SoV to guess solutions to $\frac{d}{dt}x(t) = \lambda(t)x(t)$, $x(0) = x_0 > 0$

$$\frac{d}{dt}x(t) = \lambda(t)x(t)$$

$$\frac{dx}{dt} = \lambda(t)x$$

$$\frac{dx}{x} = \lambda(t)dt$$

$$\int_{x_0}^{x(T)}\frac{dx}{x} = \int_0^T\lambda(t)dt$$

$$\ln(x(T)) - \ln(x_0) = \int_0^T\lambda(t)dt$$

$$\ln(x(T)) = \ln(x_0) + \int_0^T\lambda(t)dt$$

$$x(T) = x_0 e^{\int_0^T\lambda(t)dt}$$

Prove uniqueness

① Plug in initial cond

$$x(0) = x_0 e^{\int_0^0\lambda(t)dt} = x_0 e^0 = x_0$$

② Plug into initial diff eqn

$$\frac{d}{dt}x(t) = \frac{d}{dt}x_0 e^{\int_0^t\lambda(\tau)d\tau}$$

$$= x_0\lambda(t)e^{\int_0^t\lambda(\tau)d\tau}$$

$$= \lambda(t)x(t)$$

↳ ✓ satisfies diff eqn.

③ consider $y(t)$ that satisfies the initial cond $y(0) = x_0 \neq x_0$

④ define the ratio

$$z(t) = \frac{y(t)}{x(t)}$$

$$z(0) = \frac{y(0)}{x(0)} = \frac{x_0}{x_0} = 1$$

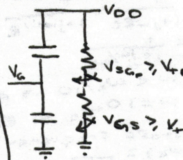
⑤ differentiate $z(t)$

$$\frac{d}{dt}z(t) = \frac{d}{dt}\frac{y(t)}{x(t)}$$

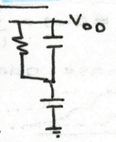
$$= 0$$

↳ implies $y(t) = x(t)$ for all t & ⇒ soln is unique

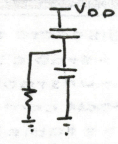
INverters

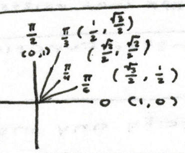


inverter outp at 1



inverter outp at 0





atan2(b,a)

b	a	θ
0	-1	$\pm\pi$
$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{5\pi}{6}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{\sqrt{2}}$	$-\frac{3\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{4\pi}{3}$
-1	0	$-\pi/2$
$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\pi}{6}$

b	a	θ
0	0	0
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\pi}{3}$
1	0	$\frac{\pi}{2}$
$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{2\pi}{3}$
$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$
0	-1	π

System ID (Fa21 #11)

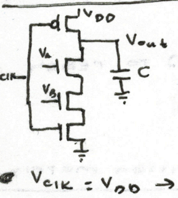
Consider a scalar system we want to model by:

$$x[i+1] = a_1 x[i] + a_2 x[i-1] + bu[i-1] + \omega[i]$$

where $\omega[i]$ is hopefully a small disturbance. We collect a trace of values for $x[i]$ and $u[i]$ for some time $i = 0, 1, \cdots, \ell$. Set up a least squares problem for the unknown parameters a_1, a_2, b in the form of solving $D\vec{p} \approx \vec{s}$.

$$\vec{s} = \begin{bmatrix} x[2] \\ \vdots \\ x[\ell] \end{bmatrix} \quad \vec{p} = \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix} \quad D = \begin{bmatrix} x[1] & x[0] & u[1] \\ \vdots & \vdots & \vdots \\ x[\ell-1] & x[\ell-2] & u[\ell-1] \end{bmatrix}$$

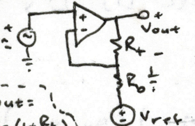
domino CMOS



V_{clk}	V_A	V_B	V_{out}
V_{DD}	0	0	V_{DD}
V_{DD}	0	V_{DD}	V_{DD}
V_{DD}	V_{DD}	0	V_{DD}
V_{DD}	V_{DD}	V_{DD}	0

- $V_{clk} = V_{DD}$ → PMOS off
  NMOS closest to grd is ON

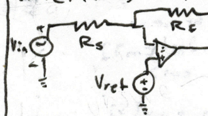
oninverting amp



$ut = n\left(1 + \frac{R_t}{R_b}\right)$

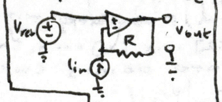
$V_{ref}(R_t/R_b)$

inverting amp



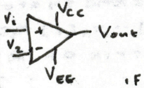
$$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right) + V_{ref}\left(\frac{R_f}{R_s} + 1\right)$$

Transresistance amp



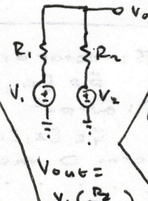
$V_{out} = -i_{in}R + V_{ref}$

Comparator



$$V_{out} = \begin{cases} V_{cc} & \text{if } V_1 > V_2 \\ V_{EE} & V_1 < V_2 \end{cases}$$

Voltage Summer



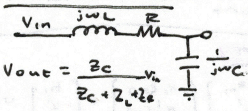
$$V_{out} = V_1\left(\frac{R_2}{R_1+R_2}\right) + V_2\left(\frac{R_1}{R_1+R_2}\right)$$

op amp golden rules

① $i_- = 0$, $i_+ = 0$

② $V_- = V_+$

LRC circuit



$$V_{out} = \frac{Z_C}{Z_C + Z_L + Z_R}V_{in}$$

$$= \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R}V_{in}$$

$$H(j\omega) = \frac{V_{out}}{V_{in}} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L + R}$$

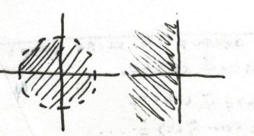
$$\rightarrow H(j\omega) = \frac{1}{1 + (j\omega)^2 LC + j\omega RC}$$

$$= \frac{1}{(j\omega)^2 + j\omega R/L + 1/LC}$$

$$\omega_n = \sqrt{1/LC}$$

↳ resonant frequency of the circuit

Stability



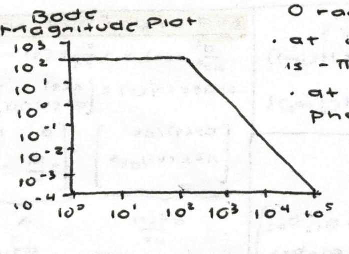
Discrete ime Stable within)

Continuous time (stable on left half)

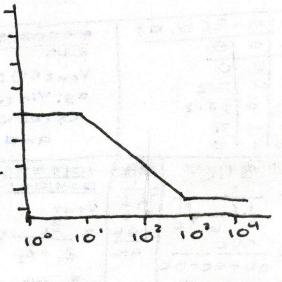
Drawing Bode Plots Based off Transfer Fcns (example)

$$H(j\omega) = \frac{100}{(1 + j\frac{\omega}{100})^2}$$

- at $\omega > \omega_c = 100$, magnitude drops by 100x per power of 10 of ω.
- at $\omega \leq \omega_c = 100$, magnitude is $A^2 = 100$



- at $\omega \leq \frac{\omega_c}{10} = 10$, phase is 0 rad = 0°
- at $\omega \geq 10\omega_c = 1000$, phase is $-\pi$ rad = -180°
- at $\omega = \omega_c = 100$ rad/s, phase is $-\frac{\pi}{2}$ rad = -90°



Complex Inner Products

$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^T}{\|\vec{u}\|^2}$$

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}\vec{u}^T}{\|\vec{u}\|^2}\vec{v} = \frac{\vec{u}^T\vec{v}}{\|\vec{u}\|^2}\vec{u} \; \Big] \text{Projection of } \vec{v} \text{ onto } \vec{u}$$

- If $\vec{v}$ is complex, we define its length or norm by

$$\|\vec{v}\|^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i\overline{v_i}$$

- If $\vec{v}$ is <u>not</u> complex, (real) we define its length/norm by

$$\|\vec{v}\|^2 = \sum_{i=1}^{n} v_i^2$$

- real inner product: $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i = \vec{u}^T\vec{v} = \vec{v}^T\vec{u}$
- complex inner product: $\underbrace{\langle \vec{u}, \vec{v} \rangle = \overline{\vec{v}}^T \vec{u} = \vec{v}^* \vec{u} = \sum_{i=1}^{n} u_i\overline{v_i}}_{\text{order matters!}}$

$$P_{\vec{u}} = \frac{\vec{u}\,\overline{\vec{u}}}{\|\vec{u}\|^2} \qquad \langle \vec{a}, \vec{b} \rangle = b^* \vec{a} = (\overline{\vec{b}})^T \vec{a} = \vec{a}^T(\overline{\vec{b}}) = \vec{a}^T(\overline{\vec{b}})$$

$$\vec{u}^* = (\overline{\vec{u}})^T$$

↪ Hermitian

controllability

State-space models

Discrete-time state space model: $\vec{x}[k+1] = A\overset{\text{state vector}}{\vec{x}}[k] + B\overset{\text{input vector}}{\vec{u}}[k]$

↳ this is controllable if, given any $\vec{x}(0)$, we can specify a series of control inputs $u[0] \cdots u[n]$ to reach any state $\vec{x}[n]$

Controllability matrix $\mathcal{C}$

$$\mathcal{C} = [\vec{b} \quad A\vec{b} \quad A^2\vec{b} \quad \cdots \quad A^{n-1}\vec{b}]$$

↳ if $\mathcal{C}$ full-rank, the system is controllable

How to reach a given state $\vec{x}[n]$, from $\vec{x}[0]$

$$\vec{x}[n] = \mathcal{C}\begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix} \Rightarrow \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix} = \mathcal{C}^{-1}\vec{x}[n]$$

↰ solve for your controls!